



## AC21007: Haskell Lecture 4

### Higher order functions, map, folds

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# Recapitulation



- ▶ Data type tuple (a, b)
- ▶ Non-strict semantics:
  - ▶ expressions evaluated on-demand
  - ▶ allows infinite data structures (lists)

# Anonymous (lambda) functions



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```
2 + 3 :: Int
2 + x ::      Int
```

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2 + 3 :: Int
2 + x ::      Int
Not in scope: 'x'
```

# Anonymous (lambda) functions



- ▶ Functions without a name
- ▶ Syntax:

$$\backslash \langle \text{var}_1 \rangle \dots \langle \text{var}_n \rangle \rightarrow \langle \text{expr} \rangle$$

Variables  $\text{var}_1$  to  $\text{var}_n$  in scope in the expression  $\text{expr}$

$$\begin{aligned} & 2 + 3 \quad :: \quad \text{Int} \\ \backslash x \rightarrow & 2 + x \quad :: \quad \text{Int} \rightarrow \text{Int} \end{aligned}$$

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- ▶ Anonymous functions:
  - ▶ can be applied to an argument:

$$(\backslash x \rightarrow 2 + x) 3 \implies 5$$
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- ▶ E.g.:

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filter pred (x:xs) = if (pred x)
  then x : filter pred xs
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- ▶ E.g:

```
filter (\x -> x `mod` 2 == 1) [1, 2, 3, 4, 5, 6]
==> [1, 3, 5]
```

```
filter (\x -> x `mod` 2 == 0) [1, 2, 3, 4, 5, 6]
==> [2, 4, 6]
```

# First-class functions



- ▶ All functions can be passed as an argument, e.g standard functions even and odd:

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filter odd [1, 2, 3, 4, 5, 6]
==> [1, 3, 5]
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- ▶ All functions are just values
- ▶ We will call functions that take a function as an argument *higher order functions*



## Some useful higher order functions

- ▶ `map` - applies a function to each element of a list

`map :: (a -> b) -> [a] -> [b]`

`map _ [] = []`

`map f (x:xs) = f x : map f xs`



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==> [2, 4, 6, 8]
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- ▶ `zipWith` - generalises `zip`, combines list elements with the function in its first argument, truncates the longer list

```
zipWith :: (a -> b -> c) -> [a] -> [b] -> [c]
```

```
zipWith _ [] _ = []
```

```
zipWith _ _ [] = []
```

```
zipWith f (a:as) (b:bs) = f a b : zipWith f as bs
```

```
zipWith (+) [2, 3, 4] [5, 6, 7]
[7, 9, 11]
```

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max x y = if x > y then x else y
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- ▶ **Note:** In a function definition all equations must have the same number of LHS patterns

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max' :: (d, d) -> d
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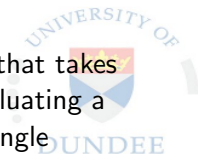
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- ▶ A variant of `max`:

$$\begin{aligned} \text{max}' &:: (d, d) \rightarrow d \\ \text{max}' (x, y) &= \text{if } x > y \text{ then } \dots \end{aligned}$$

- ▶ We can express this translation as higher-order function:

$$\begin{aligned} \text{curry} &:: ((a, b) \rightarrow c) \rightarrow a \rightarrow b \rightarrow c \\ \text{curry } f \ x \ y &= f (x, y) \end{aligned}$$

- ▶ There is also the reverse translation:

$$\begin{aligned} \text{uncurry} &:: (a \rightarrow b \rightarrow c) \rightarrow (a, b) \rightarrow c \\ \text{uncurry } f \ (x, y) &= f \ x \ y \end{aligned}$$

# Function manipulation

- ▶ Composition
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- The usual  $(f.g)(x) = f(g(x))$
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- `filter even . (filter (\ x -> x 'mod' 3 == 0))`

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- ▶ E.g: `max 5, (1 +), (2 *)`



# List folding

- ▶ Let's compare two recursive functions on lists:

- ▶ Function sum:

```
sum :: [Integer] -> Integer
sum []           = 0
sum (x : xs)    = x + sum xs
```

- ▶ Function maximum:

```
maximum :: [Integer] -> Integer
maximum (x : []) = x
maximum (x : xs) = max x (maximum xs)
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- ▶ Base case is different ...

## List folding (cont.)



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```

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```
sum   [1, 2, 3, 4, 5]
```

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## List folding - foldr and foldl



- ▶ One generic function `foldr` for right-associative recursion:

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foldr :: (a -> b -> b) -> b -> [a] -> b
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foldr f z [x1, x2, ..., xn]
  ==> f x1 (f x2 ... (f xn) ...)
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```
foldr f z [x1, x2, ..., xn]
  ==> f x1 (f x2 ... (f xn) ...)
```

- ▶ There is also function

```
foldl :: (b -> a -> b) -> b -> [a] -> b
```

for left-associative recursion, i.e.:

```
foldl f z [x1, x2, ..., xn]
  ==> f xn (... (f x2 (f x1) ...)
```

## List folding - examples



- ▶ Our sum and maximum as folds:

```
sum :: [Int] -> Int
sum xs = foldr (+) 0 xs
```

```
maximum :: [Int] -> Int
maximum []      = error "empty list"
maximum (x:xs) = foldr max x xs
```

- ▶ A fold where a and b are different:

```
length :: [a] -> Integer
length xs = foldr f 0 xs
  where
    -- f :: a -> Integer -> Integer
    f _ b = 1 + b
```

## Next time



- ▶ Monday the the 8th of February, 2-3PM, Dalhousie 3G05 LT2
- ▶ Sorting algorithms on lists
  - ▶ Selection Sort
  - ▶ Insertion Sort
  - ▶ Bubble Sort