Coinductive Soundness of Corecursive Type Class Resolution

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Research questions

- Soundness of resolution on a fragment of Horn-clause logic
  - inductive soundness w.r.t. the (standard) least Herbrand models
  - coinductive soundness w.r.t. the greatest Herbrand models
- Motivated by type-classes resolution
- Feasibility of context update techniques for corecursive type class resolution
- The fragment
  - no existential variables
  - no overlapping clauses
  - resolution by matching

Problem by an example

Example (Equality on Pairs)

```
class Eq x where
  eq :: Eq x ⇒ x → x → Bool

instance (Eq x, Eq y) ⇒ Eq (Pair x y) where
  eq (Pair x1 y1) (Pair x2 y2) = eq x1 x2 && eq y1 y2

instance Eq Int where
  eq x y = primtiveIntEq x y
```

Example (Φ_{Pair})

$$\begin{aligned} \kappa_1 &: \mathrm{eq}(x),\ \mathrm{eq}(y) &&\Rightarrow \mathrm{eq}(\mathtt{pair}(x, y)) \\ \kappa_2 &: &&\Rightarrow \mathrm{eq}(\mathtt{int}) \end{aligned}$$

$$\mathrm{eq}(\mathtt{pair}(\mathtt{int}, \mathtt{int})) \to_{\kappa_1} \mathrm{eq}(\mathtt{int}), \mathrm{eq}(\mathtt{int}) \to_{\kappa_2} \mathrm{eq}(\mathtt{int}) \to_{\kappa_2} \emptyset$$

Models

- *Herbrand base* $\mathbf{B}_{\Sigma}$ - set of all (finite) ground atoms
- *Semantic operator* $\mathcal{T}_P : 2^{\mathbf{B}_{\Sigma}} \rightarrow 2^{\mathbf{B}_{\Sigma}}$ -

$$\mathcal{T}_P(I) = \{A \in \mathbf{B}_{\Sigma} \mid B_1, \ldots B_n \Rightarrow A \text{ is a ground instance of a program clause, and } \{B_1, \ldots, B_n\} \subseteq I\}$$

- The *least Herbrand model* is the least set such that it is a fixed point of $\mathcal{T}_P$.
- The *greatest Herbrand model* is the greatest set such that it is a fixed point of $\mathcal{T}_P$.

We use $P \vDash_{ind} A$ and $P \vDash_{coind}$ to denote that A is is the least and the greatest model respectively

Type Class Resolution

Definition

$$\text{if } (e : B_1, \ldots, B_n \Rightarrow A) \in \Phi \; \frac{\Phi \vdash e_1 : \sigma B_1 \quad \cdots \quad \Phi \vdash e_n : \sigma B_n}{\Phi \vdash e \; e_1 \cdots e_n : \sigma A} \; (\textsc{Lp-m})$$

Example

$$\frac{\overline{\Phi_{Pair} \vdash \kappa_2 : \mathtt{eq}(\mathtt{int})} \qquad \overline{\Phi_{Pair} \vdash \kappa_2 : \mathtt{eq}(\mathtt{int})}}{\Phi_{Pair} \vdash \kappa_1 \kappa_2 \kappa_2 : \mathtt{eq}(\mathtt{pair}(\mathtt{int}, \mathtt{int}))}$$

Theorem

Let Φ be an axiom environment for a logic program P, and let $\Phi \vdash e : A$ hold. Then $P \vDash_{ind} A$.

Data types *Even* and *Odd*

Example ($\Phi_{EvenOdd}$)

$$\begin{aligned}
\kappa_1 &: \mathtt{eq}(x),\ \mathtt{eq}(\mathtt{evenList}(x)) &&\Rightarrow \mathtt{eq}(\mathtt{oddList}(x))\\
\kappa_2 &: \mathtt{eq}(x),\ \mathtt{eq}(\mathtt{oddList}(x)) &&\Rightarrow \mathtt{eq}(\mathtt{evenList}(x))\\
\kappa_3 &: &&\Rightarrow \mathtt{eq}(\mathtt{int})
\end{aligned}$$

$$\underline{\mathtt{eq}(\mathtt{evenList}(\mathtt{int}))} \to_{\kappa_2} \mathtt{eq}(\mathtt{int}),\ \mathtt{eq}(\mathtt{oddList}(\mathtt{int})) \to_{\kappa_3}$$
$$\mathtt{eq}(\mathtt{oddList}(\mathtt{int})) \to_{\kappa_1} \mathtt{eq}(\mathtt{int}),\ \mathtt{eq}(\mathtt{evenList}(\mathtt{int})) \to_{\kappa_3}$$
$$\underline{\mathtt{eq}(\mathtt{evenList}(\mathtt{int}))} \to_{\kappa_2} \ldots$$

Corecursive Type Class Resolution

Definition

$$\text{if HNF}(e)\ \frac{\Phi, (\alpha : \Rightarrow A) \vdash e : A}{\Phi \vdash \nu\alpha.e : A} \qquad (\textsc{Nu'})$$

Example

$$\dfrac{\dfrac{}{\kappa_3 : \texttt{eq}(\texttt{int})} \quad \vdash \kappa_3 : \texttt{eq}(\texttt{int}) \qquad \dfrac{\dfrac{}{\kappa_3 : \texttt{eq}(\texttt{int})} \vdash \kappa_3 : \texttt{eq}(\texttt{int}) \quad \dfrac{}{\alpha : \Rightarrow \texttt{eq}(\texttt{evenList}(\texttt{int}))} \vdash \alpha : \texttt{eq}(\texttt{evenList}(\texttt{int}))}{\Phi_{EvenOdd}, \alpha : \_ \vdash \kappa_1\kappa_3\alpha : \texttt{eq}(\texttt{oddList}(\texttt{int}))}}{\dfrac{\Phi_{EvenOdd}, \alpha : \_ \vdash \kappa_2\kappa_3(\kappa_1\kappa_3\alpha) : \texttt{eq}(\texttt{evenList}(\texttt{int}))}{\Phi_{EvenOdd} \vdash \nu\alpha.\kappa_2\kappa_3(\kappa_1\kappa_3\alpha) : \texttt{eq}(\texttt{evenList}(\texttt{int}))}\ \textsc{Nu'}}$$

Theorem

Let Φ be an axiom environment for a logic program P, and let F be an atomic formula. If $\Phi \vdash e : F$ by the LP-M *and* NU' *rules, then $P \vDash_{coind} F$.*

Data type *Bush*

Example (Φ_{Bush})

$$\kappa_1 : \quad \Rightarrow \mathtt{eq}(\mathtt{int})$$
$$\kappa_2 : \mathtt{eq}(x),\ \mathtt{eq}(\mathtt{bush}(\mathtt{bush}(x))) \Rightarrow \mathtt{eq}(\mathtt{bush}(x))$$

$$\mathtt{eq}(\mathtt{bush}(\mathtt{int})) \to_{\kappa_2} \mathtt{eq}(\mathtt{int}), \mathtt{eq}(\mathtt{bush}(\mathtt{bush}(\mathtt{int})) \to_{\kappa_1} \ldots \to_{\kappa_2}$$
$$\mathtt{eq}(\mathtt{bush}(\mathtt{int})), \mathtt{eq}(\mathtt{bush}(\mathtt{bush}(\mathtt{bush}(\mathtt{int}))) \to_{\kappa_1} \ldots$$

Extended Corecursive Type Class Resolution

Definition

$$\frac{\Phi, (\beta_1 : \Rightarrow B_1), \ldots, (\beta_n : \Rightarrow B_n) \vdash e : A}{\Phi \vdash \lambda\beta_1, \ldots, \beta_n.e : B_1, \ldots, B_n \Rightarrow A} \quad (\textsc{Lam})$$

$$\text{if HNF}(e) \; \frac{\Phi, (\alpha : B_1, \ldots, B_n \Rightarrow A) \vdash e : B_1, \ldots, B_n \Rightarrow A}{\Phi \vdash \nu\alpha.e : B_1, \ldots, B_n \Rightarrow A} \quad (\textsc{Nu})$$

Example

$$\dfrac{\dfrac{}{\Phi_{Bush} \vdash \kappa_1 : \mathrm{eq}(\mathtt{int})} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\Phi_{Bush}, (\alpha : \mathrm{eq}(x) \Rightarrow \mathrm{eq}(\mathtt{bush}(x))), (\beta : \Rightarrow \mathrm{eq}(x)) \vdash \kappa_2\beta(\alpha(\alpha\beta)) : \mathrm{eq}(\mathtt{bush}(x))}}{\Phi_{Bush}, (\alpha : \_) \vdash \lambda\beta.\kappa_2\beta(\alpha(\alpha\beta)) : \mathrm{eq}(x) \Rightarrow \mathrm{eq}(\mathtt{bush}(x))}\,\textsc{Lam}}{\Phi_{Bush} \vdash \nu\alpha.\lambda\beta.\kappa_2\beta(\alpha(\alpha\beta)) : \mathrm{eq}(x) \Rightarrow \mathrm{eq}(\mathtt{bush}(x))}\,\textsc{Nu}}{\Phi_{Bush} \vdash (\nu\alpha.\lambda\beta.\kappa_2\beta(\alpha(\alpha\beta)))\kappa_1 : \mathrm{eq}(\mathtt{bush}(\mathtt{int}))}$$

Soundness of Extended Corecursive Type Class Resolution

Theorem

Let Φ be an axiom environment for a logic program P, and let be $\Phi \vdash e : F$ for a formula F by the Lp-m, Lam, *and* Nu *rules. Then $P \vDash_{coind} F$.*

Proposition

There exists a logic program P, an axiom environment Φ for P and a formula F such that $\Phi \vdash e : F$ by the Lp-m, Lam, *and* Nu *rules and $P \nvDash_{ind} F$.*

Consider $\Phi_A = \{\kappa : A \Rightarrow A\}$.
Then $\Phi_A \vdash \nu\alpha.\kappa\alpha : A$ and $P_A \nvDash_{ind} A$.

Conclusion & Future Work

Conclusion

- We provide a uniform analysis of soundness of type class resolution
- We establish coinductive soundness of extended corecursive type class resolution
- We demonstrate that corecursive type class resolution is not sound inductively

Future Work

- Program transformation for extended type class resolution
- Completeness properties for extended type class resolution