Proof-Relevant Resolution for Constructive Automation

František Farka

April 23, 2019

Heriot-Watt University and University of St Andrews

Automation for Programming Languages

Type inference, term synthesis, and type classes

```
-- type inference, term synthesis
data maybe_A : Bool → Set where
  nothing : maybe_A false
  just : A → maybe_A true

fromJust : maybe_A true → A
fromJust (just x) = x
```

Type inference, term synthesis, and type classes

```
-- type inference, term synthesis
data maybe_A : Bool → Set where
  nothing : maybe_A false
  just : A → maybe_A true

fromJust : maybe_A true → A
fromJust (just x) = x
```

$$
\begin{aligned}
\mathsf{fromJust} = \lambda\ (m : \mathsf{maybe}_A\ \mathsf{true}) \to\ & \\
& \mathsf{elim}_{\mathsf{maybe}_A}\ ?_b\ m \\
& \quad (\lambda\ (w : ?_A) \to ?_e) \\
& \quad (\lambda\ (w : ?_B)\ (x : A) \to x)
\end{aligned}
$$

Type inference, term synthesis, and type classes

```
-- type inference, term synthesis
data maybe_A : Bool → Set where
  nothing : maybe_A false
  just : A → maybe_A true

fromJust : maybe_A true → A
fromJust (just x) = x
```

$$
\begin{aligned}
\mathsf{fromJust} = \lambda\ (m : \mathsf{maybe}_A\ \mathsf{true}) \to\ & \\
& \mathsf{elim}_{\mathsf{maybe}_A}\ \boxed{?_b}\ m \\
& \quad (\lambda\ (w : \boxed{?_A}) \to \boxed{?_e}) \\
& \quad (\lambda\ (w : \boxed{?_B})\ (x : A) \to x)
\end{aligned}
$$

```
-- type class resolution
class Eq a where
  eq : a → a → Bool

instance Eq Int where
  eq  x y = ...
instance (Eq a, Eq b) ⇒ Eq (a,b)
  where
  eq  (x1, x2) (y1, y2) =
    eq x1 y1 ∧ eq x2 y2

test : Eq ( Int , Int ) ⇒ Bool
test = eq (1, 2) (1, 3)
```

Type inference, term synthesis, and type classes

```
-- type inference, term synthesis
data maybe_A : Bool → Set where
  nothing : maybe_A false
  just : A → maybe_A true

fromJust : maybe_A true → A
fromJust (just x) = x
```

$$
\begin{aligned}
\mathsf{fromJust} = \lambda\ (m : \mathsf{maybe}_A\ \mathsf{true}) \to\ & \\
& \mathsf{elim}_{\mathsf{maybe}_A}\ ?_b\ m \\
& \quad (\lambda\ (w : ?_A) \to ?_e) \\
& \quad (\lambda\ (w : ?_B)\ (x : A) \to x)
\end{aligned}
$$

```
-- type class resolution
class Eq a where
  eq : a → a → Bool

instance Eq Int where
  eq  x y = ...
instance (Eq a, Eq b) ⇒ Eq (a,b)
  where
  eq  (x1, x2) (y1, y2) =
    eq x1 y1 ∧ eq x2 y2

test : Eq ( Int , Int ) ⇒ Bool
test = eq (1, 2) (1, 3)
```

```
test : Bool
test = eq { ?EqD } (1, 2) (1, 3)
```

Proof-Relevant Resolution

Big-step operational semantics

$$D := A \mid G \Rightarrow D \mid \forall x : A.D$$
$$G := A \mid D \Rightarrow G \mid \forall x : A.G \mid \exists x : A.G$$
$$e := \kappa \mid e\ e \mid \lambda\kappa.e \mid \langle M, e\rangle$$

$$\boxed{\mathcal{S}; \mathcal{P} \longrightarrow e : G}$$

$$\boxed{\mathcal{S}; \mathcal{P} \xrightarrow{e':D} e : A}$$

$$\frac{\mathcal{S}; \mathcal{P} \xrightarrow{\kappa:D} e : A \qquad \kappa : D \in \mathcal{P}}{\mathcal{S}; \mathcal{P} \longrightarrow e : A}\ \text{decide}$$

$$\frac{\mathcal{S}; \mathcal{P} \longrightarrow e : G[M/x] \qquad \mathcal{S}; \cdot \vdash M : A}{\mathcal{S}; \mathcal{P} \longrightarrow \langle M, e\rangle : \exists x : A.G}\ \exists\text{R}$$

$$\frac{}{\mathcal{S}; \mathcal{P} \xrightarrow{e:A} e : A}\ \text{init}$$

$$\frac{\mathcal{S}; \mathcal{P}, \kappa : D \longrightarrow e : G}{\mathcal{S}; \mathcal{P}, \kappa : D \longrightarrow \lambda\kappa.e : D \Rightarrow G}\ \Rightarrow\text{R}$$

$$\frac{\mathcal{S}; \mathcal{P} \longrightarrow e_1 : A_1 \qquad \mathcal{S}; \mathcal{P} \xrightarrow{ee_1:D} e_2 : A_2}{\mathcal{S}; \mathcal{P} \xrightarrow{e:A_1 \Rightarrow D} e_2 : A_2}\ \Rightarrow\text{L}$$

$$\frac{\mathcal{S}, c : A; \mathcal{P} \longrightarrow e : G[c/x]}{\mathcal{S}; \mathcal{P} \longrightarrow e : \forall x : A.G}\ \forall\text{R}$$

$$\frac{\mathcal{S}; \mathcal{P} \xrightarrow{e:D[M/x]} e_2 : A_2 \qquad \mathcal{S}; \cdot \vdash M : A_1}{\mathcal{S}; \mathcal{P} \xrightarrow{e:\forall x:A_1.D} e_2 : A_2}\ \forall\text{L}$$

Small-step operational semantics

$$\hat{e} := \kappa \mid G \mid \hat{e}\,\hat{e} \mid \langle M, \hat{e}\rangle \mid \lambda\kappa.\hat{e}$$

$$C := \bullet \mid e\ C \mid \langle M, C\rangle \mid \lambda\kappa.C$$

$$\boxed{\mathcal{S};\mathcal{P} \vdash \Gamma \mid \hat{e} \overset{\hat{e}'':D}{\rightsquigarrow} \Gamma' \mid \hat{e}'}$$

$$\frac{\mathcal{S};\Gamma \vdash \sigma : \Gamma' \qquad \mathcal{S};\Gamma' \vdash \sigma A \equiv \sigma A' : \mathsf{o}}{\mathcal{S};\mathcal{P} \vdash \Gamma \mid C\{A\} \overset{\hat{e}:A'}{\rightsquigarrow} \Gamma' \mid (\sigma C)\{\hat{e}\}}$$

$$\frac{\mathcal{S};\mathcal{P} \vdash \Gamma \mid C\{A\} \overset{\hat{e}_{\mathbf{1}}\,A_{\mathbf{1}}:D}{\rightsquigarrow} \Gamma' \mid \hat{e}}{\mathcal{S};\mathcal{P} \vdash \Gamma \mid C\{A\} \overset{\hat{e}_{\mathbf{1}}:A_{\mathbf{1}}\Rightarrow D}{\rightsquigarrow} \Gamma' \mid \hat{e}}$$

$$\frac{\mathcal{S};\mathcal{P} \vdash \Gamma, Y : A_1 \mid C\{A_2\} \overset{\hat{e}_{\mathbf{1}}:D[Y/x]}{\rightsquigarrow} \Gamma' \mid \hat{e}}{\mathcal{S};\mathcal{P} \vdash \Gamma \mid C\{A_2\} \overset{\hat{e}_{\mathbf{1}}:\forall x:A_{\mathbf{1}}.D}{\rightsquigarrow} \Gamma' \mid \hat{e}}$$

$$\boxed{\mathcal{S};\mathcal{P} \vdash \Gamma \mid \hat{e} \rightsquigarrow \Gamma' \mid \hat{e}'}$$

$$\frac{\mathcal{S};\mathcal{P} \vdash \Gamma \mid C\{A\} \overset{\kappa:D}{\rightsquigarrow} \Gamma' \mid \hat{e} \qquad \kappa : D \in \mathcal{P}}{\mathcal{S};\mathcal{P} \vdash \Gamma \mid C\{A\} \rightsquigarrow \Gamma' \mid \hat{e}}$$

$$\frac{\mathcal{S};\mathcal{P} \vdash \Gamma, Y : A \mid C\{\langle Y, G[Y/x]\rangle\} \rightsquigarrow \Gamma' \mid \hat{e}}{\mathcal{S};\mathcal{P} \vdash \Gamma \mid C\{\exists x : A.G\} \rightsquigarrow \Gamma' : A \mid \hat{e}'}$$

$$\frac{\mathcal{S};\mathcal{P}, \kappa : D \vdash \Gamma \mid C\{\lambda\kappa.G\} \rightsquigarrow \Gamma' \mid \hat{e}}{\mathcal{S};\mathcal{P} \vdash \Gamma \mid C\{D \Rightarrow G\} \rightsquigarrow \Gamma' \mid \hat{e}}$$

$$\frac{\mathcal{S};\mathcal{P} \vdash \Gamma, x : A \mid C\{G\} \rightsquigarrow \Gamma' \mid \hat{e}}{\mathcal{S};\mathcal{P} \vdash \Gamma \mid C\{\forall x : A.G\} \rightsquigarrow \Gamma' \mid \hat{e}}$$

Soundness

Theorem (Soundness)

If $\mathcal{S}; \mathcal{P} \vdash \cdot \mid G \rightsquigarrow \cdot \mid e$ *then* $\mathcal{S}; \mathcal{P} \longrightarrow e : G$.

Theorem (Generalised soundness)

If $\mathcal{S}; \mathcal{P} \vdash \cdot \mid G \rightsquigarrow \Gamma' \mid e$ *then* $\mathcal{S}; \mathcal{P} \longrightarrow e : \forall \Gamma'.G$.

$$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \rightsquigarrow \Gamma' \mid \hat{e}'$$

$$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \overset{\hat{e}_1 : D}{\rightsquigarrow} \Gamma' \mid \hat{e}'$$

$$\mathcal{S}; \mathcal{P} \longrightarrow e : G$$

$$\mathcal{S}; \mathcal{P} \overset{e_1 : D}{\longrightarrow} e : A$$

$$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \rightsquigarrow \Gamma' \mid \hat{e}'$$

$$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \overset{\hat{e}_1 : D}{\rightsquigarrow} \Gamma' \mid \hat{e}'$$

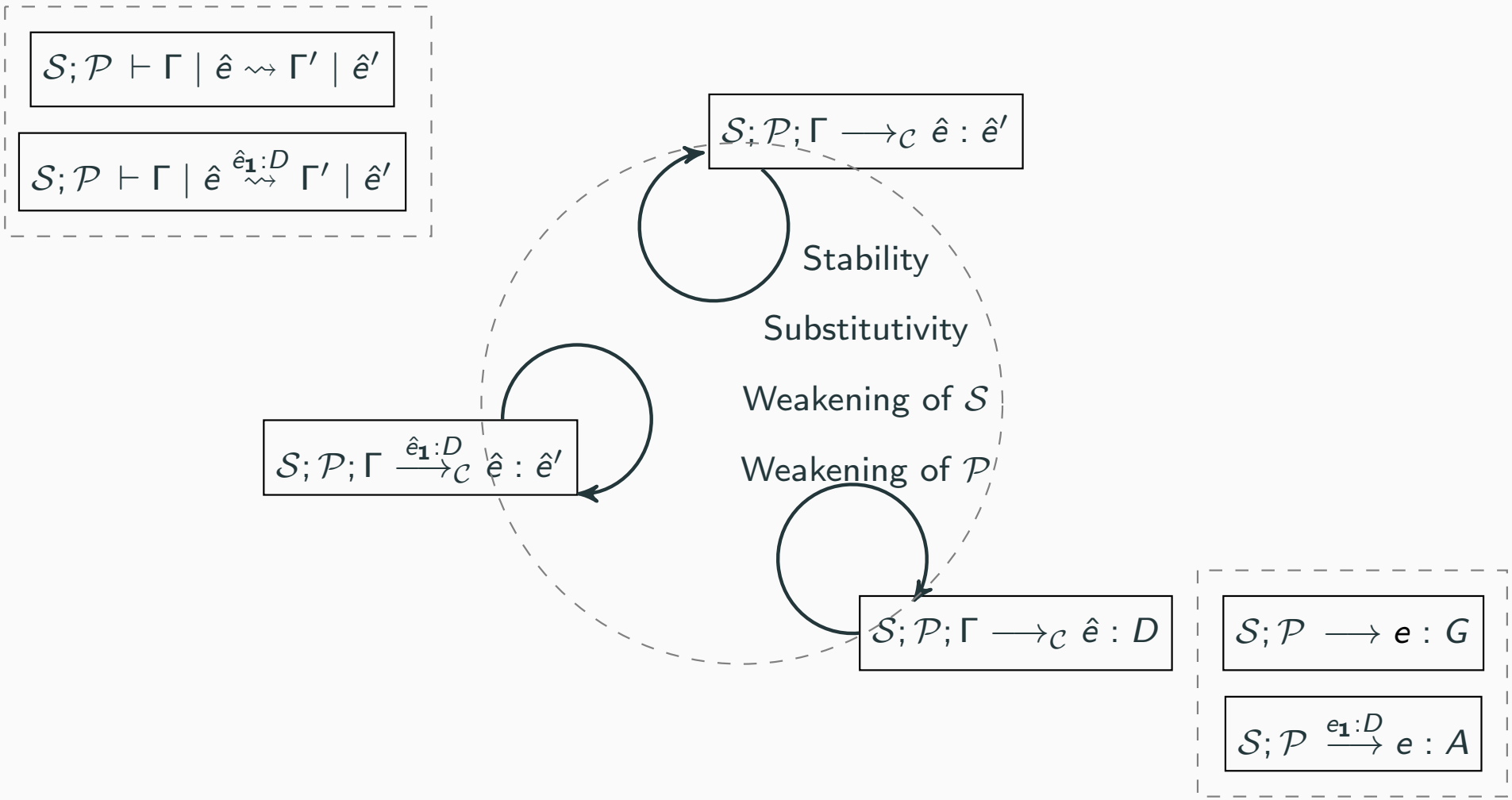
$$\mathcal{S}; \mathcal{P}; \Gamma \longrightarrow_{\mathcal{C}} \hat{e} : \hat{e}'$$

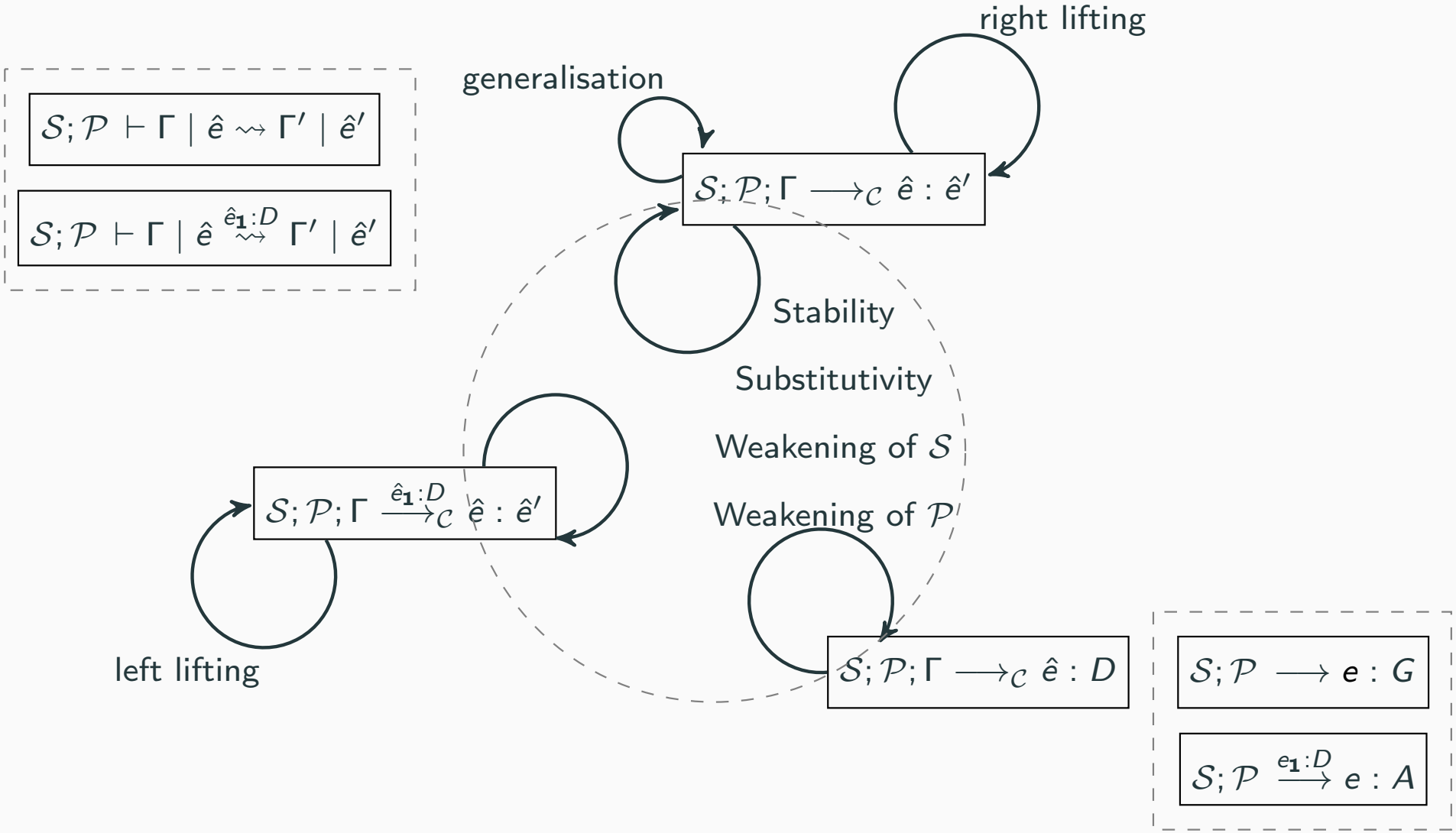
$$\mathcal{S}; \mathcal{P}; \Gamma \xrightarrow{\hat{e}_1 : D}_{\mathcal{C}} \hat{e} : \hat{e}'$$

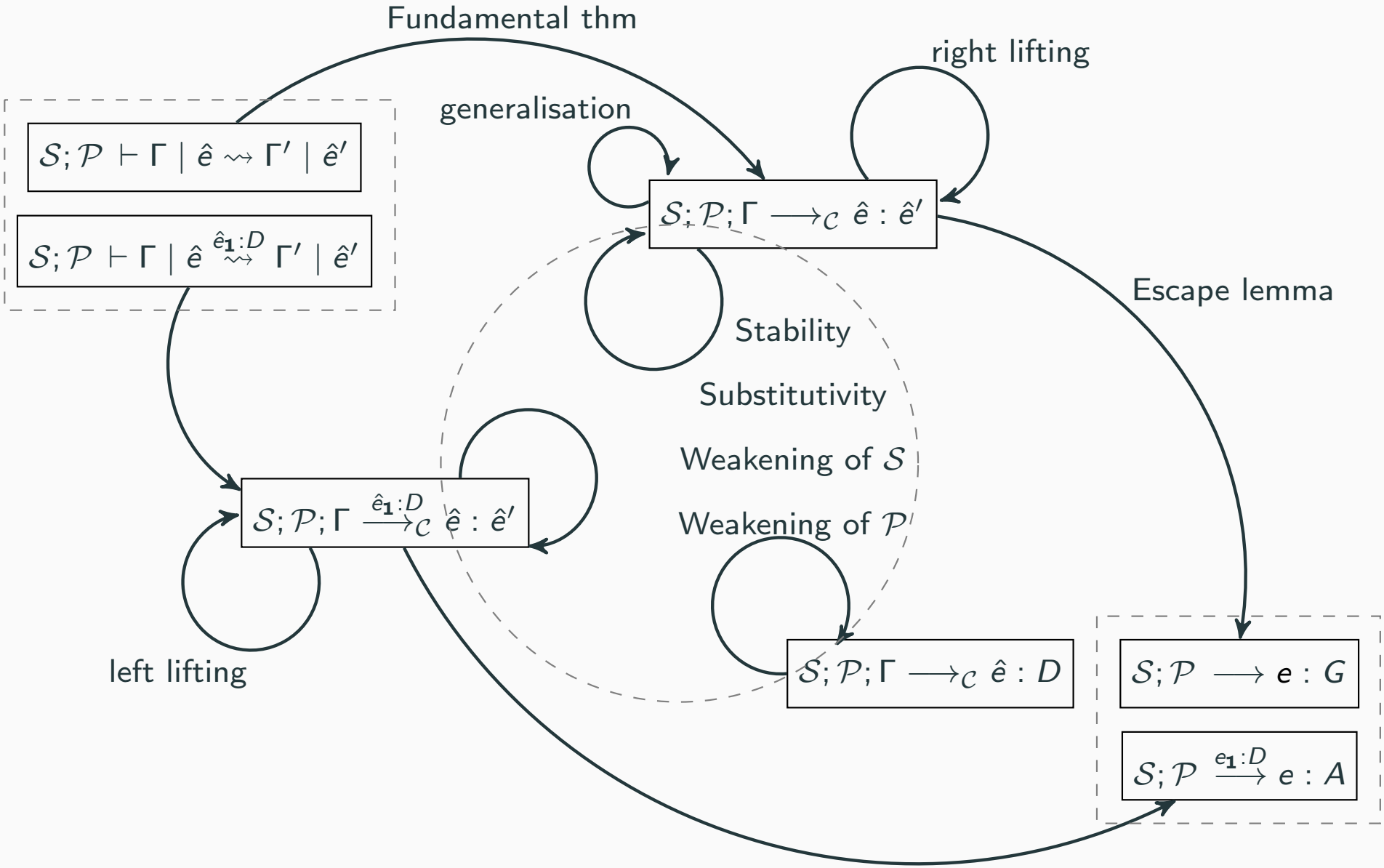
$$\mathcal{S}; \mathcal{P}; \Gamma \longrightarrow_{\mathcal{C}} \hat{e} : D$$

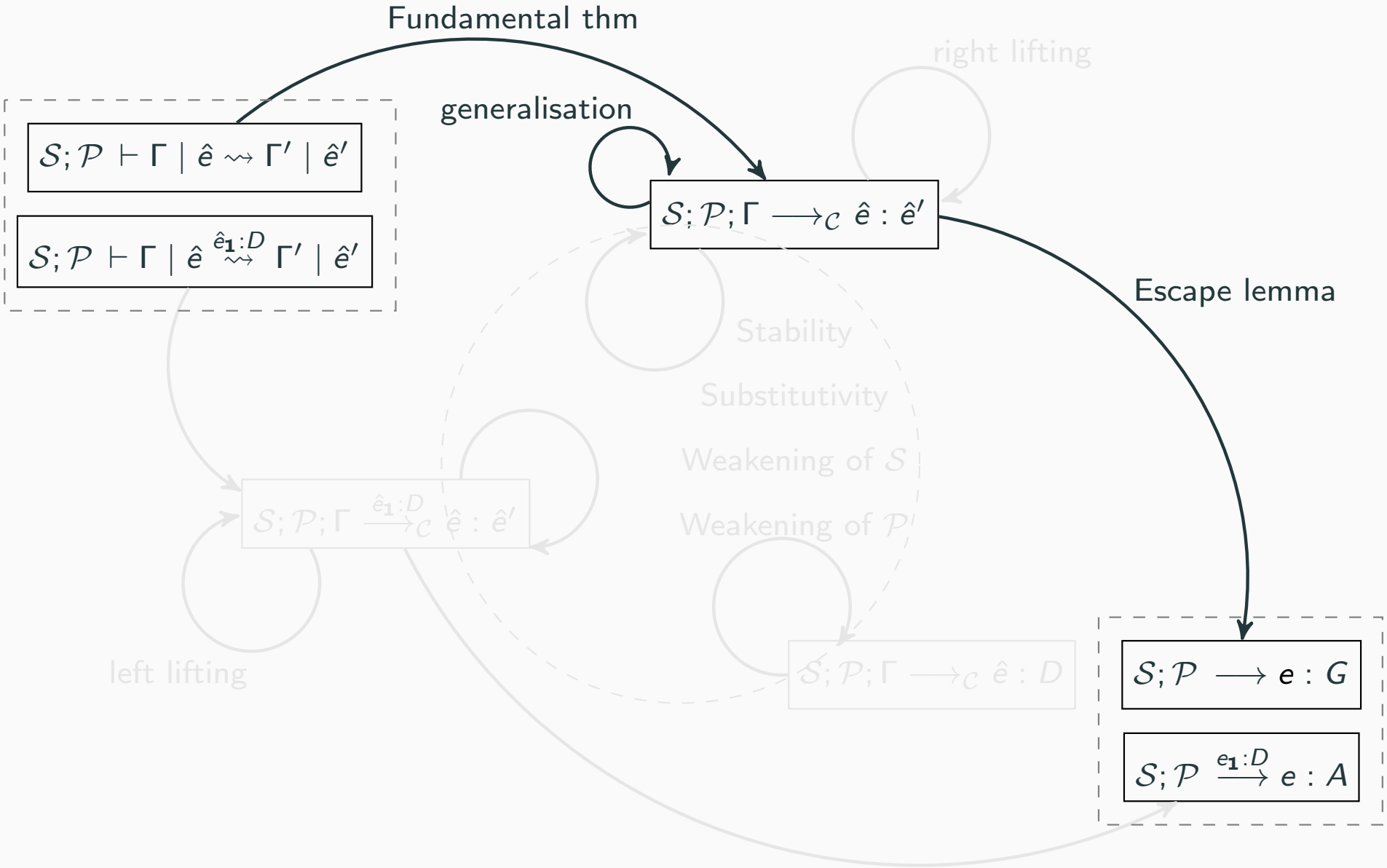
$$\mathcal{S}; \mathcal{P} \longrightarrow e : G$$

$$\mathcal{S}; \mathcal{P} \xrightarrow{e_1 : D} e : A$$


$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \rightsquigarrow \Gamma' \mid \hat{e}'$
$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \overset{\hat{e}_1 : D}{\rightsquigarrow} \Gamma' \mid \hat{e}'$
$\mathcal{S}; \mathcal{P}; \Gamma \longrightarrow_{\mathcal{C}} \hat{e} : \hat{e}'$
Stability
Substitutivity
Weakening of $\mathcal{S}$
Weakening of $\mathcal{P}$
$\mathcal{S}; \mathcal{P}; \Gamma \xrightarrow{\hat{e}_1 : D}_{\mathcal{C}} \hat{e} : \hat{e}'$
$\mathcal{S}; \mathcal{P}; \Gamma \longrightarrow_{\mathcal{C}} \hat{e} : D$
$\mathcal{S}; \mathcal{P} \longrightarrow e : G$
$\mathcal{S}; \mathcal{P} \xrightarrow{e_1 : D} e : A$

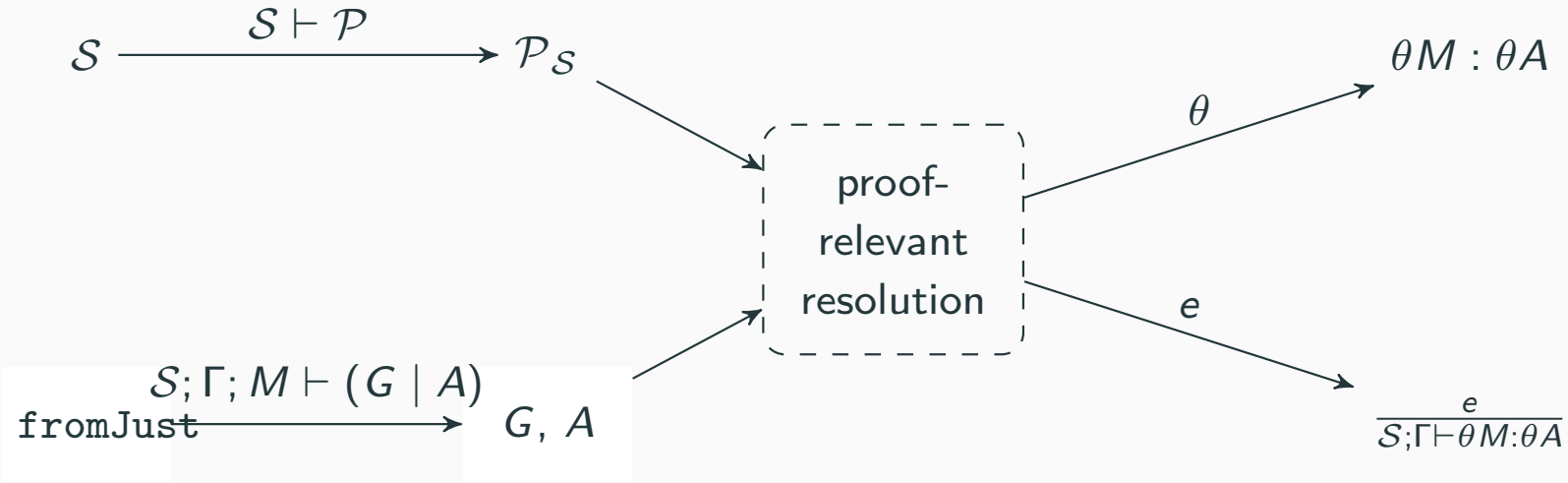

$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \rightsquigarrow \Gamma' \mid \hat{e}'$
$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \overset{\hat{e}_1 : D}{\rightsquigarrow} \Gamma' \mid \hat{e}'$
generalisation
right lifting
$\mathcal{S}; \mathcal{P}; \Gamma \longrightarrow_{\mathcal{C}} \hat{e} : \hat{e}'$
Stability
Substitutivity
Weakening of $\mathcal{S}$
Weakening of $\mathcal{P}$
$\mathcal{S}; \mathcal{P}; \Gamma \xrightarrow{\hat{e}_1 : D}_{\mathcal{C}} \hat{e} : \hat{e}'$
left lifting
$\mathcal{S}; \mathcal{P}; \Gamma \longrightarrow_{\mathcal{C}} \hat{e} : D$
$\mathcal{S}; \mathcal{P} \longrightarrow e : G$
$\mathcal{S}; \mathcal{P} \xrightarrow{e_1 : D} e : A$


Fundamental thm
right lifting
generalisation
$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \rightsquigarrow \Gamma' \mid \hat{e}'$
$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \overset{\hat{e}_1 : D}{\rightsquigarrow} \Gamma' \mid \hat{e}'$
$\mathcal{S}; \mathcal{P}; \Gamma \longrightarrow_{\mathcal{C}} \hat{e} : \hat{e}'$
Escape lemma
Stability
Substitutivity
Weakening of $\mathcal{S}$
Weakening of $\mathcal{P}$
$\mathcal{S}; \mathcal{P}; \Gamma \xrightarrow{\hat{e}_1 : D}_{\mathcal{C}} \hat{e} : \hat{e}'$
left lifting
$\mathcal{S}; \mathcal{P}; \Gamma \longrightarrow_{\mathcal{C}} \hat{e} : D$
$\mathcal{S}; \mathcal{P} \longrightarrow e : G$
$\mathcal{S}; \mathcal{P} \xrightarrow{e_1 : D} e : A$


Fundamental thm
generalisation
right lifting
$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \rightsquigarrow \Gamma' \mid \hat{e}'$
$\mathcal{S}; \mathcal{P} \vdash \Gamma \mid \hat{e} \overset{\hat{e}_1 : D}{\rightsquigarrow} \Gamma' \mid \hat{e}'$
$\mathcal{S}; \mathcal{P}; \Gamma \longrightarrow_{\mathcal{C}} \hat{e} : \hat{e}'$
Escape lemma
Stability
Substitutivity
Weakening of $\mathcal{S}$
Weakening of $\mathcal{P}$
$\mathcal{S}; \mathcal{P}; \Gamma \xrightarrow{\hat{e}_1 : D}_{\mathcal{C}} \hat{e} : \hat{e}'$
left lifting
$\mathcal{S}; \mathcal{P}; \Gamma \longrightarrow_{\mathcal{C}} \hat{e} : D$
$\mathcal{S}; \mathcal{P} \longrightarrow e : G$
$\mathcal{S}; \mathcal{P} \xrightarrow{e_1 : D} e : A$

Application to the Examples

Type inference and term synthesis[1]



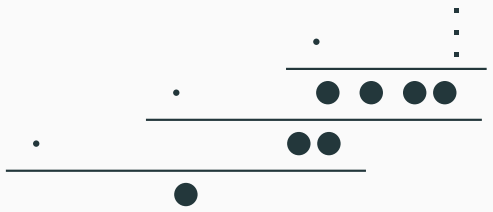
$$
\begin{aligned}
\mathsf{fromJust} = \lambda\ (m :\ \mathsf{maybe}_A\ \mathsf{true}) \to\ & \\
& \mathsf{elim}_{\mathsf{maybe}_A}\ \mathsf{true}\ m \\
& \quad (\lambda\ (w :\ \mathsf{true} \equiv \mathsf{false}) \to \mathsf{elim}_{\equiv}\ w) \\
& \quad (\lambda\ (w :\ \mathsf{true} \equiv \mathsf{true})\ (x :\ A) \to x)
\end{aligned}
$$

[1] https://github.com/frantisekfarka/slepice

Type class resolution[2]

$$\mathsf{test} = \mathsf{eq}\ \{\mathsf{eq}_{\mathsf{pair}}\ \mathsf{eq}_{\mathsf{int}}\ \mathsf{eq}_{\mathsf{int}}\}\ (1,\ 2)\ (1,\ 3)$$

$$\mathsf{data}\ \mathsf{Bush}\ a = \mathsf{Nil} \mid \mathsf{Cons}\ (\mathsf{Bush}\ (\mathsf{Bush}\ a))$$
$$\mathsf{instance}\ (\mathsf{Eq}\ a,\ \mathsf{Eq}\ (\mathsf{Bush}\ (\mathsf{Bush}\ b))) \Rightarrow \mathsf{Eq}\ (\mathsf{Bush}\ b)$$



[2] https://github.com/frantisekfarka/cotcr

Where to Next

Future work

Applications

- coinductive proof-search for parallel and distibuted computation
- constrained Horn clauses for resource-aware computation
- automation for e.g. dependently type-based probabilistic programming

Theory of proof search

different classes of sequents for efficient search space

Type inference, term synthesis, and type classes

```
-- type inference, term synthesis
data maybe_A : Bool → Set where
  nothing : maybe_A false
  just    : A → maybe_A true

fromJust : maybe_A true → A
fromJust (just x) = x

fromJust = λ (m : maybe_A true) →
        elim_maybe_A ?_b m
          (λ (w : ?_A) → ?_e)
          (λ (w : ?_B) (x : A) → x)
```

```
-- type class resolution
class Eq a where
  eq : a → a → Bool

instance Eq Int where
  eq x y = ...
instance (Eq a, Eq b) ⇒ Eq (a,b)
  where
  eq (x1, x2) (y1, y2) =
    eq x1 y1 ∧ eq x2 y2

test : Eq ( Int , Int ) ⇒ Bool
test = eq (1, 2) (1, 3)

test : Bool
test = eq { ?_EqD } (1, 2) (1, 3)
```


Big-step operational semantics

$$D := A \mid G \Rightarrow D \mid \forall x : A.D$$
$$G := A \mid D \Rightarrow G \mid \forall x : A.G \mid \exists x : A.G$$
$$e := \kappa \mid e\ e \mid \lambda\kappa.e \mid \langle M, e\rangle$$

$$\boxed{\mathcal{S};\mathcal{P} \longrightarrow e : G}$$

$$\boxed{\mathcal{S};\mathcal{P} \xrightarrow{e':D} e : A}$$

$$\frac{}{\mathcal{S};\mathcal{P} \xrightarrow{e:A} e : A}\ \text{init}$$

$$\frac{\mathcal{S};\mathcal{P} \longrightarrow e_1 : A_1 \qquad \mathcal{S};\mathcal{P} \xrightarrow{ee_1:D} e_2 : A_2}{\mathcal{S};\mathcal{P} \xrightarrow{e:A_1 \Rightarrow D} e_2 : A_2}\ \Rightarrow\text{L}$$

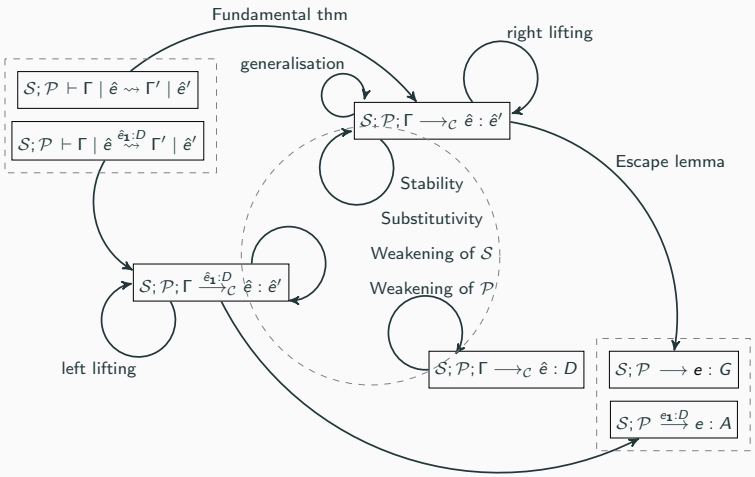
$$\frac{\mathcal{S};\mathcal{P} \xrightarrow{e:D[M/x]} e_2 : A_2 \qquad \mathcal{S};\cdot \vdash M : A_1}{\mathcal{S};\mathcal{P} \xrightarrow{e:\forall x:A_1.D} e_2 : A_2}\ \forall\text{L}$$

$$\frac{\mathcal{S};\mathcal{P} \xrightarrow{\kappa:D} e : A \qquad \kappa : D \in \mathcal{P}}{\mathcal{S};\mathcal{P} \longrightarrow e : A}\ \text{decide}$$

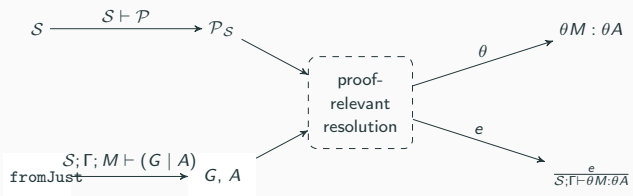
$$\frac{\mathcal{S};\mathcal{P} \longrightarrow e : G[M/x] \qquad \mathcal{S};\cdot \vdash M : A}{\mathcal{S};\mathcal{P} \longrightarrow \langle M, e\rangle : \exists x : A.G}\ \exists\text{R}$$

$$\frac{\mathcal{S};\mathcal{P},\kappa : D \longrightarrow e : G}{\mathcal{S};\mathcal{P},\kappa : D \longrightarrow \lambda\kappa.e : D \Rightarrow G}\ \Rightarrow\text{R}$$

$$\frac{\mathcal{S},c : A;\mathcal{P} \longrightarrow e : G[c/x]}{\mathcal{S};\mathcal{P} \longrightarrow e : \forall x : A.G}\ \forall\text{R}$$



Type inference and term synthesis[1]




```
fromJust = λ (m : maybe_A true) →
        elim_maybe_A true m
          (λ (w : true ≡ false) → elim_≡ w)
          (λ (w : true ≡ true) (x : A) → x)
```

[1] https://github.com/frantisekfarka/slepice

Appendix

Big-step operational semantics

Example

$$\begin{aligned}\mathcal{P} = \kappa_z &: odd(z),\\ \kappa_e &: \forall x : a.odd\ a \Rightarrow even\,(s\,x)\\ \kappa_o &: \forall x : a.even\ a \Rightarrow odd\,(s\,x)\end{aligned}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\mathcal{S}, c : a; \mathcal{P}, \kappa_x : even\ c \longrightarrow \kappa_x : even\ c}{\mathcal{S}, c : a; \mathcal{P}, \kappa_x : even\ c \longrightarrow \kappa_o\,\kappa_x : odd\,(s\,c)}}{\mathcal{S}, c : a; \mathcal{P}, \kappa_x : even\ c \longrightarrow \kappa_e\,(\kappa_o\,\kappa_x) : even\,(s\,(s\,c))}}{\mathcal{S}, c : a; \mathcal{P} \longrightarrow \lambda\kappa_x.\kappa_e\,(\kappa_o\,\kappa_x) : even\ c \Rightarrow even\,(s\,(s\,c))}}{\mathcal{S}; \mathcal{P} \longrightarrow \lambda\kappa_x.\kappa_e(\kappa_o\kappa_x) : \forall x : a.even\ x \Rightarrow even\,(s\,(s\,x))}$$

Small-step operational semantics

Example

$$\cdot \mid \forall x : a.\mathit{even}\, x \Rightarrow \mathit{even}\,(s\,(s\,x)) \rightsquigarrow \cdot \mid \mathit{even}\, c \Rightarrow \mathit{even}\,(s\,(s\,c)) \rightsquigarrow$$

$$\cdot \mid \lambda\kappa_x.\mathit{even}\,(s\,(s\,c)) \rightsquigarrow \cdot \mid \lambda\kappa_x.\mathit{even}\,(s\,(s\,c))^{\kappa_e:\forall x:a.\mathit{odd}\, x \Rightarrow \mathit{even}\,(s\,x)} \rightsquigarrow$$

$$X : a \mid \lambda\kappa_x.\mathit{even}\,(s\,(s\,c))^{\kappa_e:\mathit{odd}\, X \Rightarrow \mathit{even}\,(s\,X)} \rightsquigarrow$$

$$X : a \mid \lambda\kappa_x.\mathit{even}\,(s\,(s\,c))^{\kappa_e(\mathit{odd}\, X):\mathit{even}\,(s\,X)} \rightsquigarrow$$

$$\cdot \mid \lambda\kappa_x.\kappa_e\,(\mathit{odd}\,(s\,c)) \rightsquigarrow \cdot \mid \lambda\kappa_x.\kappa_e\,(\mathit{odd}\,(s\,c))^{\kappa_o:\forall x:a.\mathit{even}\, x \Rightarrow \mathit{odd}\,(s\,x)} \rightsquigarrow$$

$$Y : a \mid \lambda\kappa_x.\kappa_e(\mathit{odd}\,(s\,c))^{\kappa_o:\mathit{even}\, Y \Rightarrow \mathit{odd}\,(s\,Y)} \rightsquigarrow \cdot \mid \lambda\kappa_x.\kappa_e\,(\kappa_o\,(\mathit{even}\, c)) \rightsquigarrow$$

$$\cdot \mid \lambda\kappa_x.\kappa_e\,(\kappa_o\,(\mathit{even}\, c)^{\kappa_x:\mathit{even}\, c}) \rightsquigarrow \cdot \mid \lambda\kappa_x.\kappa_e\,(\kappa_o\,\kappa_x)$$

Logical relation

$$\boxed{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e} : \hat{e}'}$$

$$\frac{\mathcal{S};\mathcal{P};\Gamma \xrightarrow{\hat{e}':D}_{\mathcal{C}} \hat{e} : A \qquad \mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e}' : D}{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e} : A}$$

$$\frac{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e} : G[M/x] \qquad \mathcal{S};\Gamma \vdash M : A}{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \langle M, \hat{e}\rangle : \exists x : A.G}$$

$$\frac{\mathcal{S};\mathcal{P},\kappa : D;\Gamma \longrightarrow_{\mathcal{C}} \hat{e} : G}{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \lambda\kappa.\hat{e} : D \Rightarrow G}$$

$$\frac{\mathcal{S},c : A;\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e}[c/x] : G[c/x]}{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e} : \forall x : A.G}$$

$$\frac{\mathcal{S} \vdash \mathcal{P} \qquad \mathcal{S} \vdash \Gamma}{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} A : A}$$

$$\frac{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e}_{\mathbf{1}} : \hat{e}_{\mathbf{2}}}{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} (\theta\hat{e})\,\hat{e}_{\mathbf{1}} : \hat{e}\,\hat{e}_{\mathbf{2}}}$$

$$\frac{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e}_{\mathbf{1}} : \hat{e}_{\mathbf{2}}}{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \langle \theta M, \hat{e}_{\mathbf{1}}\rangle : \langle M, \hat{e}_{\mathbf{2}}\rangle}$$

$$\frac{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e}_{\mathbf{1}} : \hat{e}_{\mathbf{2}}}{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \lambda\kappa.\hat{e}_{\mathbf{1}} : \lambda\kappa.\hat{e}_{\mathbf{2}}}$$

$$\boxed{\mathcal{S};\mathcal{P};\Gamma \xrightarrow{\hat{e}:D}_{\mathcal{C}} \hat{e} : \hat{e}'}$$

$$\frac{}{\mathcal{S};\mathcal{P};\Gamma \xrightarrow{\hat{e}:A}_{\mathcal{C}} \hat{e} : A}$$

$$\frac{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e}_{\mathbf{1}} : A_{\mathbf{1}} \qquad \mathcal{S};\mathcal{P};\Gamma \xrightarrow{\hat{e}\,\hat{e}_{\mathbf{1}}:D}_{\mathcal{C}} \hat{e}_{\mathbf{2}} : A_{\mathbf{2}}}{\mathcal{S};\mathcal{P};\Gamma \xrightarrow{\hat{e}:A_{\mathbf{1}} \Rightarrow D}_{\mathcal{C}} \hat{e}_{\mathbf{2}} : A_{\mathbf{2}}}$$

$$\frac{\mathcal{S};\mathcal{P};\Gamma \xrightarrow{\hat{e}:D[M/x]}_{\mathcal{C}} \hat{e}_{\mathbf{2}} : A_{\mathbf{2}} \qquad \mathcal{S};\Gamma \vdash M : A_{\mathbf{1}}}{\mathcal{S};\mathcal{P};\Gamma \xrightarrow{\hat{e}:\forall x:A_{\mathbf{1}}.D}_{\mathcal{C}} \hat{e}_{\mathbf{2}} : A_{\mathbf{2}}}$$

$$\boxed{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e} : D}$$

$$\frac{\mathcal{S} \vdash \mathcal{P} \qquad \kappa : D \in \mathcal{P} \qquad \mathcal{S} \vdash \Gamma}{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \kappa : D}$$

$$\frac{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e} : A \Rightarrow D \qquad \mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e}' : A}{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e}\,\hat{e}' : D}$$

$$\frac{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e} : \forall x : A.D \qquad \mathcal{S};\Gamma \vdash M : A}{\mathcal{S};\mathcal{P};\Gamma \longrightarrow_{\mathcal{C}} \hat{e} : D[M/x]}$$

Type class resolution - Pair

Example

$\mathcal{P}_{Pair} =$

$$\begin{array}{lll} \kappa_1 : & \mathtt{eq}(x),\ \mathtt{eq}(y) & \Rightarrow \mathtt{eq}(\mathtt{pair}(x, y)) \\ \kappa_2 : & & \Rightarrow \mathtt{eq}(\mathtt{int}) \end{array}$$

$$\dfrac{\dfrac{}{\mathcal{P}_{Pair} \longrightarrow \kappa_2 : \mathtt{eq}(\mathtt{int})}\ \text{Lp-m} \qquad \dfrac{}{\mathcal{P}_{Pair} \longrightarrow \kappa_2 : \mathtt{eq}(\mathtt{int})}\ \text{Lp-m}}{\mathcal{P}_{Pair} \longrightarrow \kappa_1\kappa_2\kappa_2 : \mathtt{eq}(\mathtt{pair}(\mathtt{int}, \mathtt{int}))}\ \text{Lp-m}$$

Type class resolution - Bush

Example

$$\mathcal{P}_{Bush} = \begin{array}{ll} \kappa_1 : & \Rightarrow \mathtt{eq}(\mathtt{int}) \\ \kappa_2 : & \mathtt{eq}(x),\ \mathtt{eq}(\mathtt{bush}(\mathtt{bush}(x))) \Rightarrow \mathtt{eq}(\mathtt{bush}(x)) \end{array}$$

$$\dfrac{\dfrac{}{\begin{array}{c}\mathcal{P}_{Bush} \longrightarrow \\ \kappa_1 : \mathtt{eq}(\mathtt{int})\end{array}} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots \\ \mathcal{P}_{Bush}, (\alpha : \mathtt{eq}(x) \Rightarrow \mathtt{eq}(\mathtt{bush}(x))), (\beta : \ \Rightarrow \mathtt{eq}(x)) \longrightarrow \\ \kappa_2\beta(\alpha(\alpha\beta)) : \mathtt{eq}(\mathtt{bush}(x))\end{array}}{\mathcal{P}_{Bush}, (\alpha : \_) \longrightarrow \lambda\beta.\kappa_2\beta(\alpha(\alpha\beta)) : \mathtt{eq}(x) \Rightarrow \mathtt{eq}(\mathtt{bush}(x))}\ \text{Lam}}{\mathcal{P}_{Bush} \longrightarrow \nu\alpha.\lambda\beta.\kappa_2\beta(\alpha(\alpha\beta)) : \mathtt{eq}(x) \Rightarrow \mathtt{eq}(\mathtt{bush}(x))}\ \text{Nu}}{\mathcal{P}_{Bush} \longrightarrow (\nu\alpha.\lambda\beta.\kappa_2\beta(\alpha(\alpha\beta)))\kappa_1 : \mathtt{eq}(\mathtt{bush}(\mathtt{int}))}$$